Modification of the van Driest Damping Function to Include the Effects of Surface Roughness

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A prediction method for rough surfaces has been given that is based on a simple extension of the van Driest damping function. The method stimulates the turbulent shear stresses near the wall by manipulating the amount of viscous damping applied. It is demonstrated that the model reproduces the shift in the logarithmic layer found for sand roughness. For very high surface roughness, an intermediate logarithmic layer is found that links the viscous sublayer to the fully developed logarithmic layer. The method has been applied to a series of test cases of varying complexity and surface roughness, showing that the predictions of flows along rough surfaces can be computed with the same degree of accuracy as for flows over smooth surfaces.

Introduction

THE development of boundary layers on rough surfaces is highly dependent on the characteristics of the surface. For different surfaces with the same average heights of the roughness elements, the flow may behave very differently because the aerodynamic characteristics of the surfaces are different. The roughness elements may be small, sharp edged, and densely packed or large with well-rounded corners and vice versa. Even for surfaces covered with a single, well-defined type of roughness element, the surface distribution may be varied in an unlimited number of ways. The way the different surfaces differ in an aerodynamic sense will be in the amounts of local separations and vorticity that the surface produces. Thus, close to the surface where the flow is dominated by the local roughness elements, the velocity distribution would have to be a function of a large number of geometry-dependent variables k_1 to k_n defining the surface texture. The flow will also depend on the magnitude of the wall shear stress τ_w and fluid properties such as the fluid density ρ and viscosity μ

$$U = F(\rho,\mu,\tau_w,y,k_1,\ \ldots,\ k_n) \tag{1}$$

It is not likely that a unique relation will exist that may characterize a rough surface aerodynamically from general statistics of the grain size. Until such a relation eventually can be developed, we will assume that the aerodynamic characteristics of the surface may be related to a reference surface that may be characterized by a single roughness length. We take this to be a surface completely covered with sand grain roughness. (It should be mentioned here that the prediction method of Taylor, Coleman, and Hodge[1] in fact predicts the near-surface flow by considering the local drag of individual roughness elements. However, this requires a detailed knowledge of the roughness distribution that is not compatible with the common knowledge of engineering types of surfaces.) Dirling[2] related the surface height, density, and inclination to an equivalent sand grain roughness, but a large degree of uncertainty still exists in the determination of the equivalent sand grain roughness for an arbitrary roughness geometry. (See the review by Young and Paterson.[3]) However, from simple experiments for a given surface, the equivalent sand roughness is easily obtained.

In the region close to the surface elements, a strong mechanism for generating turbulence will exist if the local Reynolds number is sufficiently high. A commonly used criterion for defining the surface as aerodynamically rough is that the dimensionless roughness height is $k^+ > 5$. This means that the roughness elements protrude up through the viscous sublayer. It may then be assumed that if this turbulence production is caused by local separations from the individual roughness elements, the eddy size should be comparable to the size of the element itself. Therefore it may be expected that close to the surface the mixing length may be considerably larger than the value produced at a smooth wall, i.e., $l^+ = \kappa y^+$.

At larger distances from the wall, the flow will be fully turbulent, and we will not know what mechanism generated the turbulence. We may therefore assume that except for a constant that depends on the inner region, the velocity in this region will show the same dependence as that for a smooth surface. This region is therefore normally expressed as

$$U^+ = \frac{1}{\kappa} \ln y^+ + A - \Delta U^+(k^+) \tag{2}$$

where the change in level of the curve due to roughness is expressed as

$$\Delta U^+ = \frac{1}{\kappa} \ln k^+ + B \tag{3}$$

here the superscript denotes variables that have been made dimensionless using wall variables. These are the friction velocity $U_\tau = \sqrt{\tau_w/\rho}$ and the viscous length scale ν/U_τ so that $U^+ = U/U_\tau$ and $k^+ = kU_\tau/\nu$. The A is the constant that exists for a smooth surface.

From this we see that a turbulence model for rough surfaces must generate a velocity profile that has a logarithmic region with the same slope at large distances from the wall for all roughness scales. It is also necessary that the velocity shift compared to the smooth surface is the shift that is found experimentally.

A number of turbulence models have been proposed that allow the calculation of turbulent boundary layers on rough surfaces by minor modifications to models for smooth surfaces. Common to most of them is that they are based on the work by Rotta.[4] Rotta argued that the direct influence of surface roughness is only felt very close to the surface. He then came to the conclusion that this effect on the law of the wall could be obtained by shifting the reference surface plane a distance $-\Delta y^+$ with respect to the mean roughness level. To satisfy the no-slip condition at the new surface, the reference plane would have to move in the direction opposite to the

Received Sept. 5, 1989; revision received March 26, 1990.

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mean flow at a rate that depends on this shift. Although this produces the desired effect on the law of the wall, it does not seem to have any direct physical relation to the problem since the shift required is frequently much higher than the height of the roughness element itself.

Cebeci and Chang[5] included the idea of Rotta in their mixing length model by deriving a curve fit to the Δy^+ distribution specified by Rotta. In the mixing length estimate, the effective wall distance was then taken as $y^+ + \Delta y^+$ while the no-slip condition was maintained at $y^+ = 0$. Hoekstra,[6] introducing the shift in the reference plane in a manner similar to that of Cebeci and Chang, managed to derive simple analytical functions relating Δy^+ to k^+. A very similar relation was derived by Shkvar.[7]

Chan[8] successfully included the effect of roughness by applying the idea of a reference plane slip velocity. However, he did not use the shift in reference plane position. Apparently no prediction method exists that has fully included all aspects of Rotta's ideas.

Turbulence Model

The main effect of surface roughness is to increase the turbulence close to the roughness elements. The primary goal for a turbulence model is therefore to incorporate this effect in the way the turbulent shear stresses are modeled by relating this increase directly to the surface roughness. The basis of the proposed model is the van Driest[9] damping function combined with the mixing length formulation of Michel, Quémard, and Durant.[10]

The velocity distribution in the wall region may then be written[11]

$$U^+ = \int_0^{y^+} \frac{2\mathrm{d}y^+}{1 + \sqrt{1 + (2l^+F)^2}} \tag{4}$$

where l^+ is the dimensionless mixing length and F the damping function. This shows that if the product l^+F grows linearly with respect to y^+ when this is significantly larger than 1, we are guaranteed that U^+ will have a region of logarithmic growth. The slope of this region is given by the inverse of the constant that relates l^+F to y^+. The formulation used for l^+ is the model of Michel et al.[10]

$$l^+ = \frac{0.085\delta U_\tau}{\nu} \tanh\left(\frac{\kappa y}{0.085\delta}\right) \tag{5}$$

where the von Kármán constant κ is taken to be $\kappa = 0.41$. A damping function was first proposed by van Driest who wrote it

$$F = 1 - \exp\left(-\frac{y^+}{A^+}\right) + \exp\left(-\frac{y^+R^+}{A^+k^+}\right) \tag{6}$$

When using the mixing length model for the turbulent stresses, the total shear stress is highly overestimated close to the wall. The effect of the damping function is to reduce the contribution of the turbulent stress for y^+ distances less than about $3A^+$. With the commonly accepted value of $A^+ = 26$, it is seen that the flow will be undamped from about $y^+ = 75$.

In the case of a rough surface, the extent of the viscous layer close to the wall is reduced since the turbulent mixing is more vigorous. Van Driest suggested that this effect could be obtained by reducing the amount of damping. Therefore the last term was added to Eq. (6). The R^+ was given the value of 60, which is roughly where the damping disappears. Thus we see that for $k^+ = R^+$ there is no damping so this would be the definition of a fully rough surface. Obviously the same cancellation could be obtained by taking (R^+/k^+) to any power. Sivykh[12] pointed out that Eq. (6) considerably underpredicts the shift ΔU^+ in the law of the wall for high roughness numbers as shown in Fig. 1. This is because the formulation of Eq. (6) is limited to $F \le 1$. For large roughness elements, it will be necessary to increase the estimated shear stress above the corresponding undamped value for a smooth surface. (The method proposed by Granville[13] suffers from the same shortcomings. He dropped the last term in Eq. (6) and coupled the roughness directly to the value of A^+ so that this ranged from 26 to a theoretical minimum of 0.)

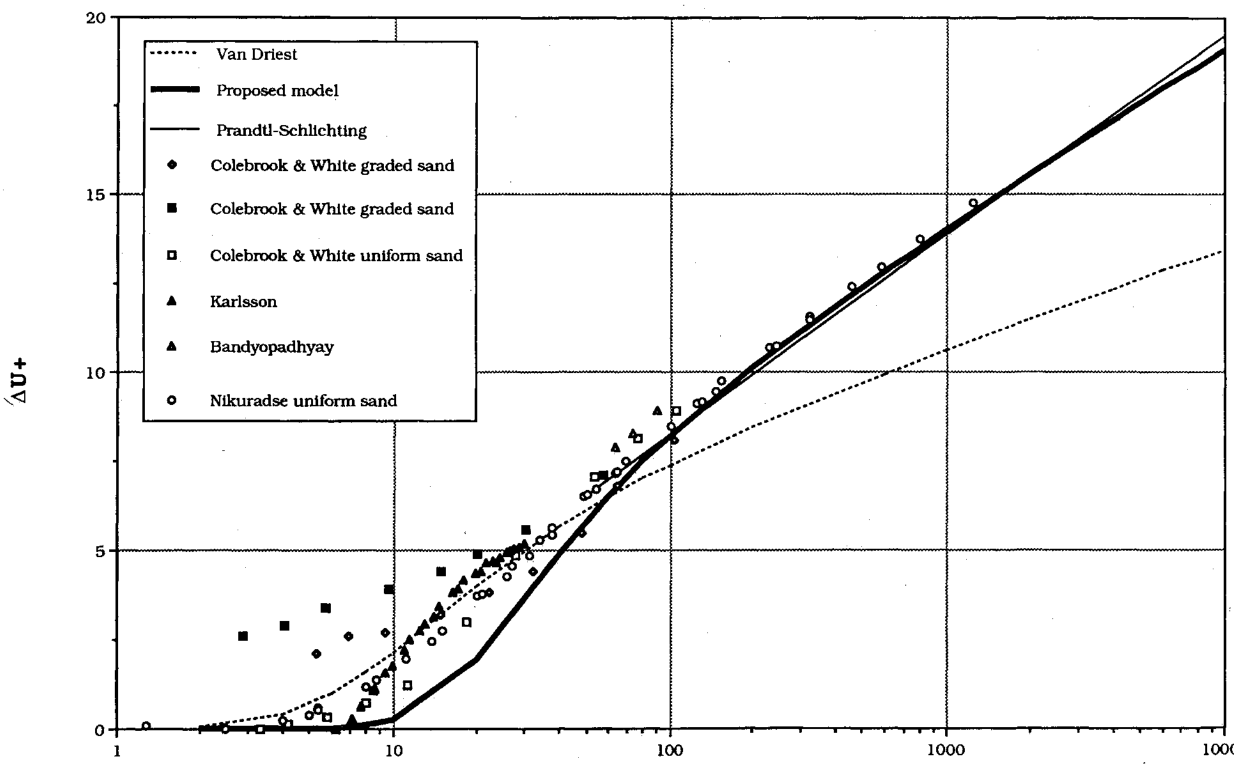

Fig. 1 Shift in the law of the wall due to roughness [predictions using Eqs. (6) and (7) compared to experiments[15–18]].

To reproduce the correct shift, the following formulation is proposed:

$$F = 1 - \exp\left(-\frac{y^+}{A^+}\right) + \exp\left[-\frac{y^+}{A^+}\left(\frac{R^+}{k^+}\right)^{3/2}\right]\sqrt{1+\exp\left(-\frac{R^+}{k^+}\right)} \quad (7)$$

Here $R^+ = 70$, which is the value specified by Rotta for the onset of completely rough surfaces. It also agrees better with what was stated about A^+ than the value used originally by van Driest. The chosen value produces a logarithmic increase in ΔU^+ for $k^+ > 100$. By adding the square root term, the slope is increased by the necessary amount for high values of k^+. As seen from Fig. 1 this formulation gives a considerable improvement to the van Driest formulation since it produces a velocity shift that is identical to the distribution proposed by Prandtl and Schlichting[14] based on experiments with sand roughness.

For $k^+ \leq 100$, the experimental results are no longer unique since they will depend on the fraction of elements that are sufficiently small to be considered as aerodynamically smooth ($k^+ \leq 5$). This is demonstrated by the measurements of Colebrook and White[15] shown in Fig. 1. In addition to uniform sand grain roughness, they performed measurements for different combinations of large and small sand grains. It may be seen that these data deviate for low roughness values but tend towards a common curve with the rest of the data as k^+ becomes large. No effort has therefore been made to match any particular distribution in this region. Also included are the data of Karlsson,[16] Bandyopadhyay,[17] and Nikuradse.[18] The data of Karlsson are unique since the skin friction was measured directly using a skin friction balance.

In the region where ΔU^+ grows logarithmically, Eq. (7) produces a shift that is given by

$$\Delta U^+ = \frac{1}{\kappa} \ln k^+ - 3.37 \quad (8)$$

in agreement with the recommendation of Prandtl and Schlichting.

Figure 2 shows the law of the wall that the model produces for different values of k^+. Also included in Fig. 2 is a set of zero pressure gradient measurements taken on sand grain covered surfaces. The data taken by Karlsson are plotted using the measured C_f and the tabulated velocities directly. For the other experiments, C_f was not obtained by direct measurements. The most common way to obtain C_f is by means of the method devised by Perry and Joubert.[19] Since neither C_f nor the effective probe position with respect to the wall are known, C_f and the wall position are adjusted until the best fit to the law of the wall is obtained. In doing so, a logarithmic distribution at a constant slope is assumed. As seen from Fig. 2, this is not generally the case as the logarithmic region must connect to the linear viscous sublayer. Therefore the data must be matched to the complete law of the wall. In doing so a difference of more than 10% compared to the method of Perry and Joubert is easily obtained. However, common to the chosen sets of measurements are that they have reached a state of self-similarity. For all experiments, the displacement-, momentum-, and boundary-layer thicknesses grow almost linearly. From an integral balance, the skin friction coefficient, which then is practically constant, is easily obtained and was normally given by the authors. Using this value, the wall reference plane was adjusted only slightly to match the predicted law of the wall. It should be noted that for all measurements this shift was much smaller than the shift applied by the originators to match the constant slope distribution.

As the roughness increases, the onset of the logarithmic region shifts to higher values of y^+ since this region cannot exist for $y^+ < k^+$.

For smooth surfaces the velocity close to the surface increases in a manner that contains two asymptotes: the viscous

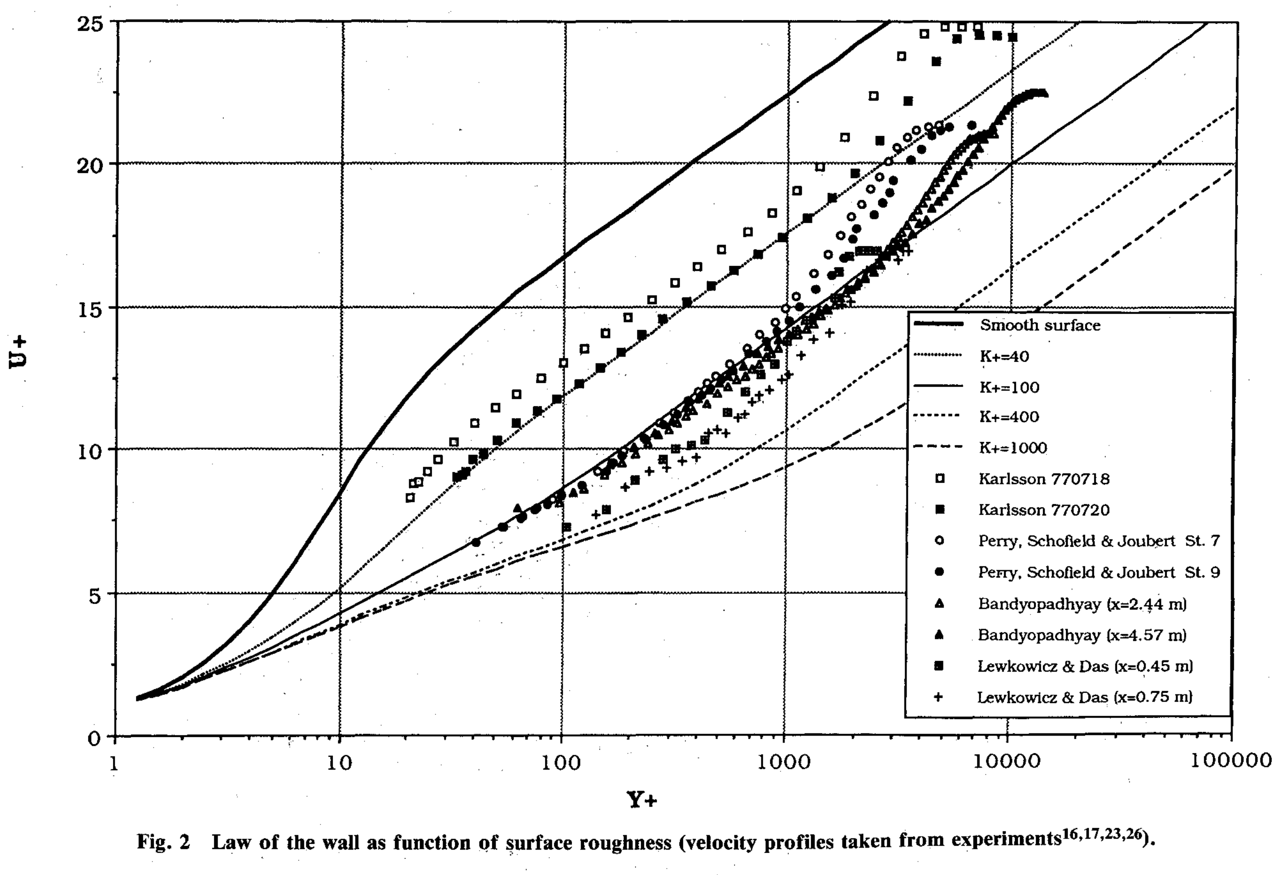

Fig. 2 Law of the wall as function of surface roughness (velocity profiles taken from experiments[16,17,23,26]).

sublayer and the fully turbulent region. As k^+ becomes large, the two curves no longer intersect. For very high surface roughness, i.e., for $k^+ \gg A^+$, it may be shown that both Eqs. (6) and (7) lead to a third asymptote that links the other two together. Using the latter, it may be seen that if y^+ is very small, F reduces to the constant $F=\sqrt{2}$. Then if $y^+ < (1/2\sqrt{2}\kappa) \approx 1$, Eq. (4) reduces to the linear relation for the viscous sublayer

$$U^+ = y^+ \tag{9}$$

In the intermediate range $A^+ < y^+ < k^+$, the damping function is roughly $F = 1 + \sqrt{2}$, and Eq. (4) is reduced to

$$U^+ = \int_0^{y^+} \frac{dy^+}{l^+F} \tag{10}$$

giving

$$U^+ = \frac{1}{(1+\sqrt{2})\kappa} \ln y^+ + \text{const} \tag{11}$$

that is, a logarithmic layer with a slope about 40% of the logarithmic layer found on smooth surfaces. For the highest values of k^+ in Fig. 2, this asymptote may be seen, and this behavior is also found in the data. Since this layer develops for high values of k^+, it must be related to the logarithmic part of the velocity shift. It is associated with the increased turbulence production due to the large roughness elements. So far only very few experiments exist at such high values ($k^+ > 1000$).

Finally we find that for $y^+ \gg k^+$ the damping function reduces to its fully turbulent value $F = 1$, leading to the well-known logarithmic layer

$$U^+ = \frac{1}{\kappa} \ln y^+ + \text{const} \tag{12}$$

Results

The proposed model was incorporated into the finite-difference method of Krogstad,[20] which is a fully three-dimensional prediction method for incompressible boundary layers using the ADI (alternating-direction-implicit) solution scheme.

To verify the model, it is necessary to apply it to a number of test cases of different complexities. A review of available experiments has been given by Uram.[21]

Flat Plate Flows

The flat plate test cases of Scottron and Power,[22] who performed measurements on a surface covered with a very coarse wire mesh, and of Lewkowicz and Das,[23] who did measurements on replicas of a full-scale ship surface, as well as the high Reynolds number experiment on sand grain surface roughness by Bandyopadhyay, have been computed. These experiments cover a large range of k-type surface roughness length scales using different types of roughness elements.

The mesh screen of Scottron and Power had a wire diameter of 2.67 mm and a mesh size of 12.7 mm. From the measured velocity profiles, the shift in the logarithmic region could be measured. Comparing this to Eq. (8), the equivalent sand roughness was estimated to be $k = 11.3$ mm. This is in close agreement with the equivalent sand roughness of $k = 11.9$ mm found by Cebeci and Chang in a slightly different way. The estimated value for k gives the very high dimensionless roughness height of $k^+ = 1375$ at the position where the calculations were started. The equivalent sand roughness of the Lewkowicz and Das surface was found to be $k = 2.8$ mm, giving an intermediate value of $k^+ = 298$ at the initial station. For the sand grain experiment of Bandyopadhyay, the sand grain size was found to be $k = 0.73$ mm, giving $k^+ = 92.5$ at the initial station.

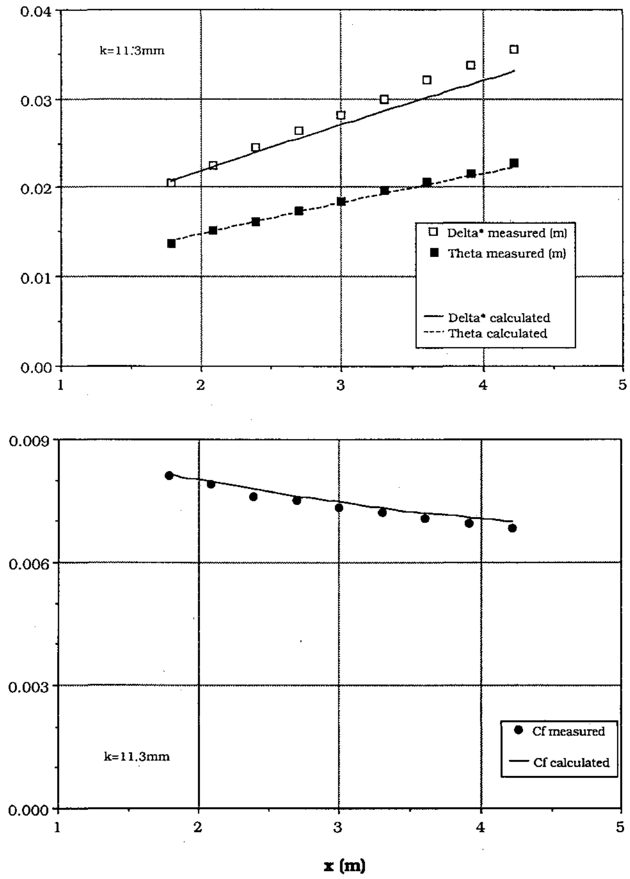

Fig. 3 Constant pressure flow of Scottron and Power[22] (geometry 0).

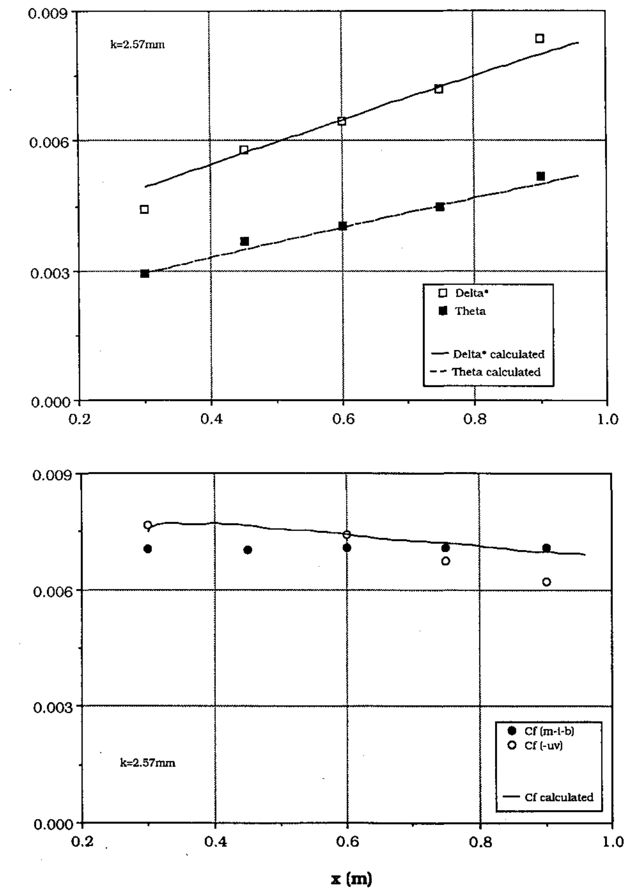

Fig. 4 Constant pressure flow of Lewkowicz and Das.[23] Skin friction was estimated from the momentum integral balance [Cf (m-i-b)] and turbulent shear stresses [Cf (-uv)].

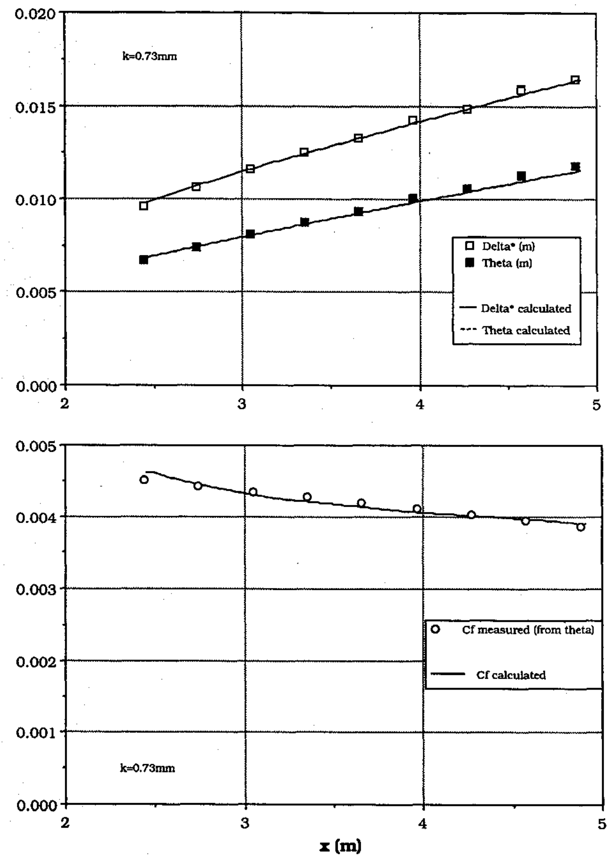

Fig. 5 Constant pressure flow of Bandyopadhyay.[17] Skin friction was estimated from the momentum integral balance.

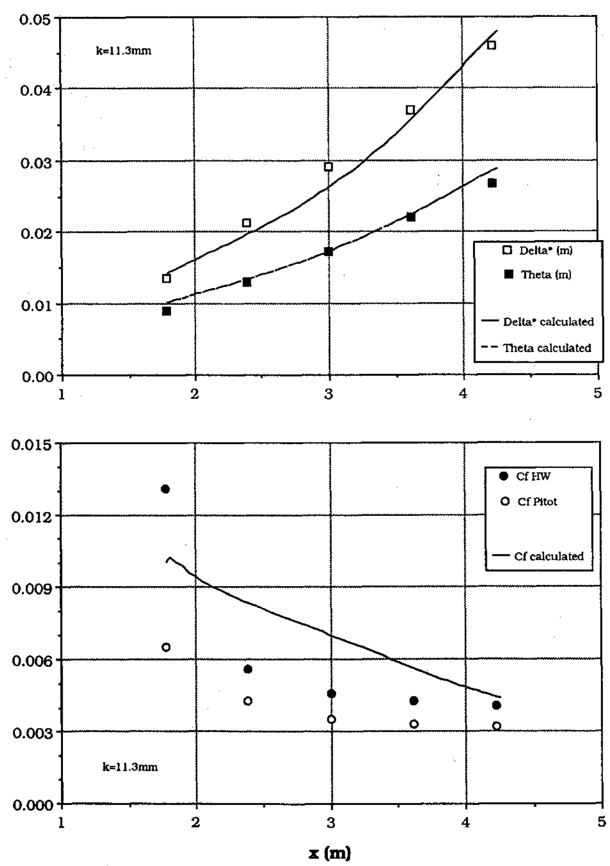

Fig. 6 Mild adverse pressure gradient flow of Scottron and Power[22] (geometry 1). Skin friction estimated from the law of the wall using mean velocity profiles measured with Pitot tube (C_f Pitot) and hot wire (C_f HW). (Perry and Joubert[19] method.)

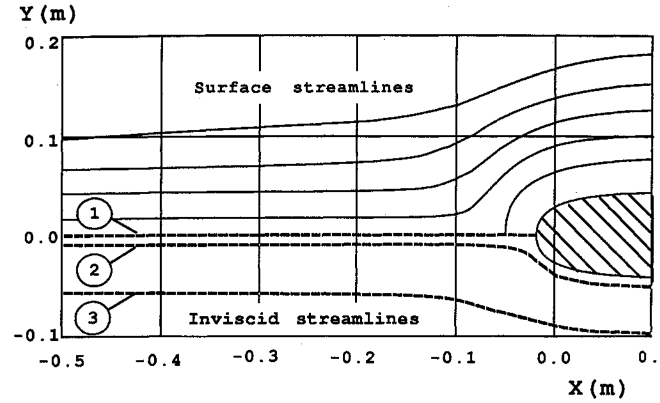

Fig. 7 Layout of the wing body junction experiment of Krogstad[24] and Krogstad and Fanneløp.[25]

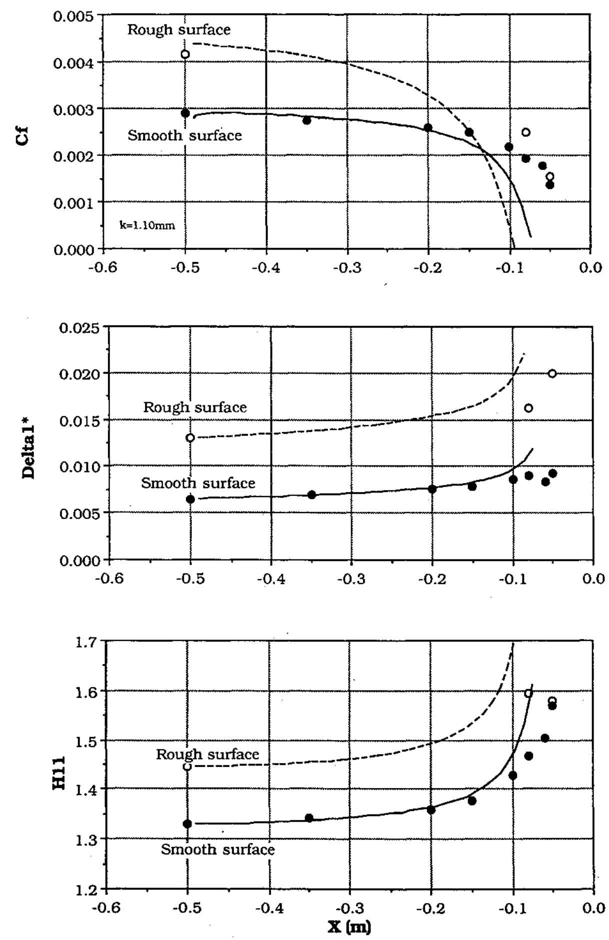

Fig. 8 Results for the wing-body junction experiment along the line of symmetry.

The results are shown in Figs. 3, 4, and 5. In none of the experiments was the skin friction coefficient measured directly. Scottron and Power calculated C_f by graphical differentiation of the measured momentum thickness distribution. Lewkowicz and Das computed C_f from a momentum integral balance. They also measured the turbulent shear stress profiles from which an estimate of C_f may be obtained. In the experiment of Bandyopadhyay, the skin friction coefficient obtained by curve-fitting and derivating the momentum thickness distribution has been shown. For all experiments, the calculations are in close agreement with the experimental data despite the large range in k^+. The displacement and momentum thicknesses are satisfactorily predicted, although the growth rate of

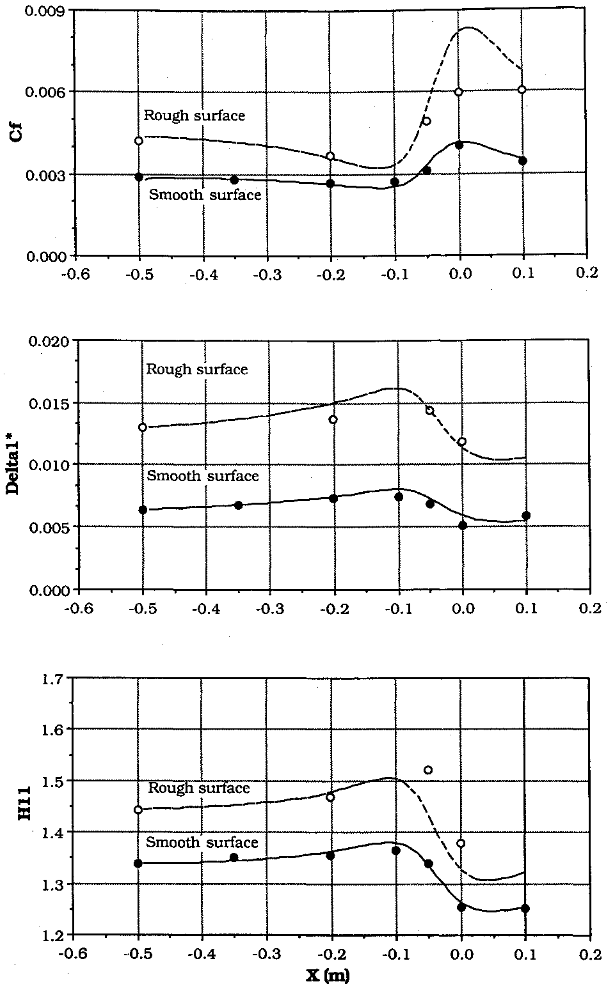

Fig. 9 Results for the wing-body junction experiment along streamline 3.

the displacement thickness for the experiment of Scottron and Power is slightly underpredicted.

Adverse Pressure Gradient Flows

Scottron and Power also made experiments with adverse pressure gradients. The results for their mild adverse pressure gradient are shown in Fig. 6. The agreement is not quite satisfactory. The predicted value of C_f in particular is too high. However, the presented results are close to the results presented by Cebeci and Chang. As the estimates of C_f from the mean velocity profiles taken by Pitot-static tube and hot wire deviate considerably, it is reasonable to believe that these data may be somewhat unreliable.

Three-Dimensional Flow

The last case calculated was the three-dimensional wing body junction boundary-layer experiment of Krogstad[24] and Krogstad and Fanneløp.[25] For this case the inviscid flow is given analytically. Measurements were taken along the three inviscid streamlines indicated in Fig. 7. The first line (line 1) is the line of symmetry that leads up to an ordinary separation point. Line 2 also leads up to the horseshoe vortex where three-dimensional separation exists, whereas line 3 passes outside the separated region.

The measurements were done twice, once on a smooth surface and once with the surface covered with commercial no. 24 sandpaper. In this way the effects of roughness could be isolated since any imperfections in the experiment such as interactions, shortcomings in the flow quality, or other effects not included in the prediction method should be the same for both surfaces. In the same way as described earlier, the equivalent sand roughness was estimated to be 1.1 mm, giving $k^+ = 72$ at the initial station.

The results are presented in Figs. 8 and 9 along line 1 and line 3. The agreement with the experimental points is seen to be about the same for both surfaces. In both the rough and smooth surface predictions, the separation point on the line of symmetry is predicted somewhat early, which is to be expected when no viscous-inviscid interaction is accounted for. For the predictions along line 3 where the pressure gradients are much smaller and very little interaction is to be expected, the agreement is seen to be very good. However, the predicted C_f along the rough surface responds somewhat too strongly to the favorable pressure gradient in the range $-0.1 < x < 0.0$.

Conclusions

A prediction method for rough surfaces has been given that is based on a simple extension of the van Driest damping function. The method differs from previous methods since it stimulates the turbulent shear stresses near the wall by manipulating the amount of viscous damping rather than introducing a shift in the location of the wall. It is demonstrated that for completely rough surfaces an intermediate logarithmic layer exists that links the fully turbulent logarithmic region to the viscous sublayer. This region is important if skin friction coefficient is to be obtained by matching velocity profiles to the law of the wall.

The method has been applied to a series of test cases of varying complexity and surface roughness, showing that the predictions for flows along rough surfaces can be computed with the same degree of accuracy as for flows over smooth surfaces.

Some deviations from the experiments may be observed in the results for the two-dimensional adverse pressure gradient flow. However, the method behaves well for the flat-plate flows and the much more complex three-dimensional flow, covering a large range of effective surface roughness. It is therefore believed that the shortcomings must be due to effects not included in the prediction method rather than in the way the roughness is handled.

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